CSE 20 Spring 2024
Practice for Test 2

Below are the instructions that will be on the first page of the test package:

INSTRUCTIONS — READ THIS NOW

- Write your name, PID, current seat number, exam time, and the academic integrity pledge in the indicated space above and on the designated **answer sheet**. We will check for **all** of this identifying information before grading. Write your answers in the specified areas, or your work will not be graded.

- We will not be answering questions about the exam during the exam period. If any bugs are found in the exam after the exam period, the affected question part(s) will be addressed.

- You may use one 8.5"x11", doublesided sheet of notes that you create and bring to the exam room, but no other books, notes, or aids.

- You may not speak to any other student in the exam room while the exam is in progress (including after you hand in your own exam). You may not share **any information** about the exam with anyone who has not taken it.

- Turn off and put away all cellphones, calculators, and other electronic devices. You may not access any electronic device during the exam period. If you need to leave the room during the exam period, you must leave all electronic devices with an exam proctor.

- To receive full credit, your answers must be written neatly, legibly, and sufficiently darkly to scan well in the indicated answer box. Your solution will be evaluated both for correctness and clarity. Read the instructions for each part carefully to determine what is required for full credit. This test has ?? problems worth a total of ?? points.

- This exam is **45 minutes** long. Read all the problems first before you start working on any of them, so you can manage your time wisely.

- Please stay seated until the end of the exam period. We will collect all exams and note sheets at the **end** of the exam period, to minimize disruption for students who wish to use the full time for the exam. Please show your ID to a proctor when asked.

1. **Predicates and Quantifiers** Express each of the following statements symbolically using quantifiers, variables, propositional connectives, and predicates (make sure to define the domain and the meaning of any predicates you introduce). Then, express its negation as a logically equivalent compound proposition without a $\neg$ in front. Decide whether the statement or its negation is true, and prove it.

    (a) If m is an integer, $m^2 + m + 1$ is odd.

    (b) For all integers n greater than 5, $2^n - 1$ is not prime.

    (c) If n and m are even integers, then $n - m$ is even.

2. **Proof strategies** Prove each of the following claims. Use only basic definitions and general proof strategies in your proofs; do not use any results proved in class / the textbook as lemmas. You may use basic properties of real numbers that we stated (without proof) in class, e.g. that the difference of two integers is an integer.

    (a) The difference of any rational number and any irrational number is irrational.

    (b) If a, b, c are integers and $a^2 + b^2 = c^2$, then at least one of a and b is even.

    (c) For any integer n, $n^2 + 5$ is not divisible by 4.

    (d) For all positive integers a, b, c, if $a \nmid bc$ then $a \nmid b$. (The notation $a \nmid b$ means "a does not divide b").

    Bonus: Watch the video `https://www.youtube.com/watch?v=MhJN9sByRSO` and write down the statement and one or both of the proofs described using the notation, definitions, and proof strategies from CSE 20.

3. **Sets** Prove or disprove each of the following statements.

    (a) For all sets A and B, $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.

    (b) For all sets A and B, $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

    (e) For any sets A, B, C, D, if the Cartesian products $A \times B$ and $C \times D$ are disjoint then either A and C are disjoint or B and D are disjoint (or both).

    (f) There are sets A, B such that $A \in B$ and $A \subseteq B$.

    (g) For all sets A, B, C: $A \cap B = \emptyset$ and $B \cap C = \emptyset$ if and only if $(A \cap B) \cap C = \emptyset$.

4. **Induction and Recursion** Prove the following statements:

    (a) $\forall s \in S\,(\,rnalen(s) = basecount(s, \mathtt{A}) + basecount(s, \mathtt{C}) + basecount(s, \mathtt{U}) + basecount(s, \mathtt{G})\,)$ where S RNA strands and the functions $rnalen$ and $basecount$ are define recursively as:

$$
\begin{array}{lll}
 & & rnalen : S \quad \to \mathbb{Z}^+ \\
\text{Basis Step:} & \text{If } b \in B \text{ then} & rnalen(b) \quad = 1 \\
\text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B, \text{ then} & rnalen(sb) \quad = 1 + rnalen(s)
\end{array}
$$

$$basecount : S \times B \ \rightarrow \mathbb{N}$$

Basis Step: If $b_1 \in B, b_2 \in B$ $\quad basecount(b_1, b_2) \ = \begin{cases} 1 & \text{when } b_1 = b_2 \\ 0 & \text{when } b_1 \neq b_2 \end{cases}$

Recursive Step: If $s \in S, b_1 \in B, b_2 \in B$ $\quad basecount(sb_1, b_2) \ = \begin{cases} 1 + basecount(s, b_2) \\ \qquad \text{when } b_1 = b_2 \\ basecount(s, b_2) \\ \qquad \text{when } b_1 \neq b_2 \end{cases}$

(b) $\forall l_1 \in L \, \forall l_2 \in L \, (sum(concat(l_1, l_2)) = sum(l_1) + sum(l_2))$, where the function $concat$ is defined as:

$$concat : L \times L \ \rightarrow L$$

Basis Step: If $l \in L$ $\quad concat([\,], l) \ = l$
Recursive Step: If $l, l' \in L$ and $n \in \mathbb{N}$, then $\quad concat((n, l), l') \ = (n, concat(l, l'))$

and the function $sum : L \to \mathbb{N}$ that sums all the elements of a list and is defined by:

$$sum : L \ \rightarrow \mathbb{N}$$

Basis Step: $\quad sum([\,]) \ = 0$
Recursive Step: If $l \in L, n \in \mathbb{N}$ $\quad sum((n, l)) \ = n + sum(l)$

5. **Induction and Recursion** At time 0, a particle resides at the point 0 on the real line. Within 1 second, it divides into 2 particles that fly in opposite directions and stop at distance 1 from the original particle. Within the next second, each of these particles again divides into 2 particles flying in opposite directions and stopping at distance 1 from the point of division, and so on. Whenever particles meet they annihilate (leaving nothing behind). How many particles will there be at time $2^{11} - 1$? You do not need to justify your answer. Hint: Derive a formula for the number/ locations of particles at time $2^n - 1$ for arbitrary positive integer n, prove the formula using induction, and apply it when $n = 11$.

6. **Induction and Recursion** Prove that every positive integer has a base 3 expansion. *Hint: use strong induction.*

7. **Functions & Cardinalities of sets** Prove each of the following claims.

(a) For the "function" $f : \mathbb{Z} \to \{0, 1, 2, 3\}$ given by $f(x) = x \bmod 5$, it is not the case that every element of the domain maps to exactly one element of the codomain (that is, it is not a well-defined function).

(b) The "function" $f : \mathcal{P}(\mathbb{Z}^+) \to \mathbb{Z}$ given by

$$f(A) = \text{the maximum element in A}$$

is not well-defined.

(c) There is a one-to-one function with domain $\{a, b, c\}$ and codomain $\mathbb{R}$.

(d) There is an onto function with domain R and codomain $\{\pi, \frac{1}{17}\}$, where R is the set of user ratings in a database with 5 movies.

(e) The Cartesian product $\mathbb{Z}^+ \times \{a, b, c\}$ is countable.

(f) The interval of real numbers $\{x \in \mathbb{R} \mid 5 \leq x \leq 8\}$ is uncountable. *Hint: you may use the fact that the unit interval $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ is uncountable.*